Also solved by H. A. LEVY, A. M. HARDING, J. A. COLSON, LEVI S. SHIVELY, H. C. FEEMSTER, ELMER SCHUYLER, and the PROPOSER.

**391. Proposed by C. N. SCHMALL, New York, N. Y.**

Show that the roots of the quadratic

$$ax^2 + 2bx + c = 0$$

are imaginary if $a$, $b$, $c$ are in harmonic progression and have the same sign.

SOLUTION BY S. W. REAVES, University of Oklahoma.

Since $a$, $b$, $c$ are in harmonic progression,

$$b = \frac{2ac}{a+c}.$$

The discriminant of the given quadratic then becomes

$$b^2 - ac = \left(\frac{2ac}{a+c}\right)^2 - ac = -ac\left(\frac{a-c}{a+c}\right)^2,$$

which is negative since $a$ and $c$ have the same sign. The discriminant being negative, the roots are imaginary.

Also solved by WALTER C. EELLS, ELMER SCHUYLER, F. M. MORGAN, H. C. FEEMSTER, J. A. COLSON, and A. M. HARDING.

Additional solutions of 387 were received from R. M. MATHEWS, HORACE OLSON, and G. Y. SOSNOW, after the December issue had gone to press.

*Note.* We have solutions of all problems proposed during 1913 in this section up to the September number, except 385, which was published in February.

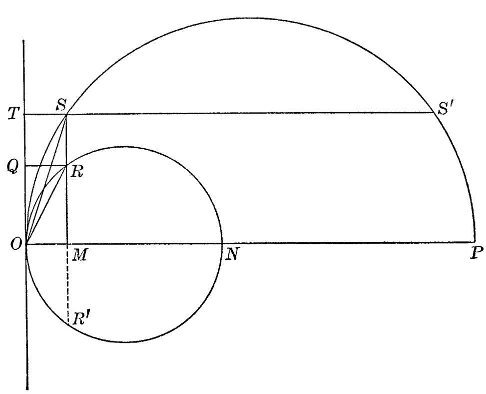
GEOMETRY.

**419. Proposed by S. LEFSCHETZ, University of Nebraska.**

Given a circle and a tangent to it. To find a point on its circumference such that the sum of its distances to the tangent and its point of contact shall be equal to a given length.

SOLUTION BY S. W. REAVES, University of Oklahoma.

Let $ORN$ be the given circle, $OT$ the given tangent, and $O$ the point of contact. Denote the given length by $l$.



*Construction.* Draw the diameter $ON$ and extend it beyond $N$ to $P$ making $NP = 2l$. On $OP$ as diameter construct the semi-circle $OSS'P$. On the given tangent lay off $OT = l$. Through $T$ draw the line $TSS'$ parallel to $OP$ to meet the semi-circumference first in $S$. Drop a perpendicular from $S$ to $OP$ meeting $OP$ in $M$ and meeting the given circle in $R$ and $R'$. Then $R$ (or $R'$) is the required point.

*Proof.* Let $QR = OM = x$, $OR = y$, $OS = z$, $ON = 2r$. Then by construction $OP = 2(r + l)$ and $MS = l$.

By elementary geometry $OM : OR = OR : ON$.

Hence

$$y^2 = 2rx. \tag{1}$$

Similarly $$OM : OS = OS : OP.$$

Hence, $$z^2 = 2x(r + l), \tag{2}$$

or, from (1), $$z^2 = y^2 + 2lx. \tag{3}$$

But $z^2 = x^2 + l^2$, and therefore (3) becomes

$$y = \pm (l - x). \tag{4}$$

Since $y > 0$ and $l > x$, the lower sign is inadmissible. Hence $x + y = l$, or

$$QR + OR = l.$$

A solution exists when and only when $l \leqq 4r$, and there is not more than one solution on the same side of the diameter through the point of contact.

Solutions were also received from WALTER C. EELLS, H. C. FEEMSTER, C. N. SCHMALL, ELMER SCHUYLER, and A. H. HOLMES.

**420. Proposed by C. N. SCHMALL, New York City.**

Four spheres are described so that each touches a face of a given triangular pyramid and the other three faces produced. If the radii of the escribed spheres be $r_1$, $r_2$, $r_3$, $r_4$, and $r$ be the radius of the inscribed sphere, show that

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} = \frac{2}{r}.$$

SOLUTION BY HORACE OLSON, Chicago, Illinois.

Let $ABCD$ be a given tetrahedron.

Let the faces $ABC$, $ABD$, $ACD$, and $BCD$ be represented by small letters corresponding to the opposite vertices of the tetrahedron, *i. e.*, by $d$, $c$, $b$, and $a$ respectively.

Let $V$ represent the volume of the tetrahedron $ABCD$.

Let $O$ be the center, and $r$ the radius, of the inscribed sphere.

Let $O_1$, $O_2$, $O_3$, and $O_4$ be the centers, and $r_1$, $r_2$, $r_3$, and $r_4$ the radii, of the escribed spheres opposite to $A$, $B$, $C$, and $D$, respectively.

Connect points $O$ and $O_3$ with the 4 vertices of the tetrahedron.

Then,

$$V = AOBC + OBCD + OACD + OABD = \frac{ar}{3} + \frac{br}{3} + \frac{cr}{3} + \frac{dr}{3} = \frac{r}{3}(a + b + c + d),$$